

Cycles in enhanced hypercubes

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Abstract

The enhanced hypercube $Q_{n,k}$ is a variant of the hypercube Q_n . We investigate all the lengths of cycles that an edge of the enhanced hypercube lies on. It is proved that every edge of $Q_{n,k}$ lies on a cycle of every even length from 4 to 2^n ; if k is even, every edge of $Q_{n,k}$ also lies on a cycle of every odd length from $k+3$ to 2^n-1 , and some special edges lie on a shortest odd cycle of length $k+1$.

Key words: interconnection network; hypercube; enhanced hypercube; cycle

1 Introduction

As a topology for an interconnection network of a multiprocessor system, the hypercube structure is a widely used and well-known interconnection model since it possesses many attractive properties [15, 23]. In particular, Q_n is vertex-transitive and edge-transitive. There are different variations of hypercubes, for example, folded hypercubes [1], balanced hypercubes [8], and augmented hypercubes [7]. These variations support efficient embedding, reduced diameter, and improve fault tolerance in comparison to the hypercube.

The enhanced hypercube $Q_{n,k}$ was introduced by Tzeng and Wei [16] to achieve improvement in diameter, mean internode distance, and traffic density. Different properties of enhanced hypercubes were investigated by now. The diagnosability of it under different models was studied in [17, 18]. To measure the fault-tolerance of them, the super spanning connectivity and super spanning laceability were determined in [2]. The authors constructed independent spanning trees in [20]. Paths and cycles embedding with faulty elements were investigated in [10, 11, 12, 14]. These results indicate that enhanced hypercubes keep numerous desirable properties of hypercubes.

A cycle structure, which is a fundamental topology for parallel and distributed processing, is suitable for local area networks and for the development of simple parallel algorithms with low communication cost. Studies of various networks about the cycle embedding problems can be found in the literature [3, 4, 13, 19]. To the best of our knowledge, there are no results about cycle embedding with respect to a specific edge in $Q_{n,k}$. The enhanced hypercube $Q_{n,k}$ is vertex transitive [20], but not edge transitive [16] except for $Q_{n,n}$ which is the folded hypercube FQ_n [22]. As can be seen from the proofs, the embedding even cycles containing a special edge is straightforward when compared with the more interesting case of embedding the shortest odd cycles.

The rest of this paper is organized as follows. In the next section we introduce the basic notation and terminology. The main results are given in Section 3.

2 Preliminaries

Throughout this paper, we follow Xu [23] for graph-theoretical terminology and notation not defined here.

A uv -walk W in a graph G is a sequence of vertices in G , beginning with u and ending at v such that consecutive vertices in the sequence are adjacent. That is, we can express W as $W = \langle u = v_0, v_1, \dots, v_l = v \rangle$ where $l \geq 0$ and $(v_i, v_{i+1}) \in E(G)$ for $0 \leq i \leq l-1$. Each vertex v_i ($0 \leq i \leq l$) and each edge (v_i, v_{i+1}) ($0 \leq i \leq l-1$) is said to lie on W . The number of edges in a walk is called the length of the walk. If $u = v$, then the walk is closed; if $u \neq v$, then W is open. A uv -path P in a graph G is a uv -walk in which no vertices are repeated. A closed walk that repeats no vertices, except for the first and last, is a cycle. The length of any cycle is at least 3. We usually express a cycle as $C = P_1 + \langle v_0, v_1, \dots, v_i \rangle$ where P_1 is a v_0v_i -path disjoint with path $\langle v_0, v_1, \dots, v_i \rangle$ except for the two vertices v_0 and v_i .

The n -dimensional hypercube Q_n is a graph with 2^n vertices, each vertex with a distinct binary string $u_n u_{n-1} \dots u_1$ on the set $\{0, 1\}$. Two vertices are linked by an edge if and only if their strings differ in exactly one bit. The edge joining the vertices differ in the i -th ($1 \leq i \leq n$) bit is called an i -dimensional edge. The Cartesian product $G \times H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. For our purposes it is essential to recall that Q_n can be represented as $Q_n = K_2 \times Q_{n-1}$.

As a variant of the hypercube, the n -dimensional folded hypercube FQ_n , proposed

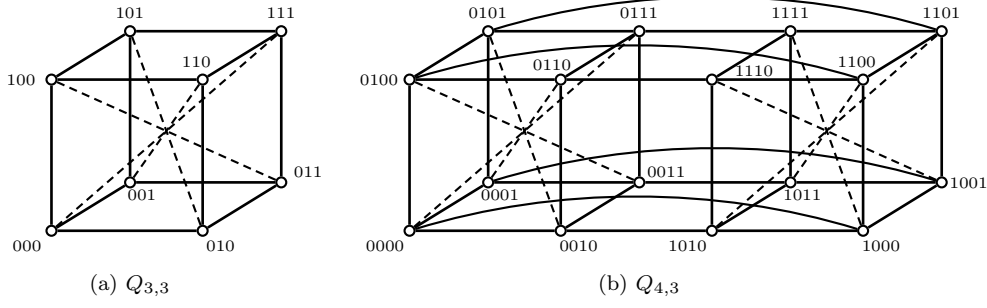


Figure 1: Enhanced hypercubes $Q_{3,3}$ and $Q_{4,3}$. The hypercube edges and skips are represented by solid lines and dashed lines, respectively.

first by El-Amawy and Latifi [1], can be obtained from the hypercube Q_n by adding an edge, called a *complementary edge*, between any pair of complementary vertices $u = u_n \dots u_2 u_1$ and $\bar{u} = \bar{u}_n \dots \bar{u}_2 \bar{u}_1$, where $\bar{u}_i = 1 - u_i$ for $i = 1, 2, \dots, n$.

The n -dimensional enhanced hypercube $Q_{n,k}$ ($1 \leq k \leq n$) is obtained from the hypercube Q_n by adding a complementary edge between two vertices $u = u_n \dots u_2 u_1$ and $v = u_n \dots u_{k+1} \bar{u}_k \bar{u}_{k-1} \dots \bar{u}_1$. We call these complementary edges skips, and denote by E_s . To distinguish E_s from the edges in Q_n , we call edges in Q_n hypercube edges and denote the set of i -dimensional hypercube edges by E_i for $i = 1, 2, \dots, n$. So the complete edge set $E(Q_{n,k})$ of a enhanced hypercube can be expressed as $E(Q_n) \cup E_s$. For instance, $Q_{3,3}$ and $Q_{4,3}$ are shown in Fig. 1.

When $k = 1$, $Q_{n,1}$ reduces to the n -dimensional hypercube Q_n . When $k = n$, $Q_{n,n}$ is the n -dimensional folded hypercube FQ_n . Since there are many results about cycle embedding in the hypercubes Q_n and folded hypercubes FQ_n [5, 6, 9, 21, 22]. We consider cycles in the enhanced hypercubes $Q_{n,k}$ where $2 \leq k \leq n - 1$ in the next section.

From the definitions of Cartesian product graph and the enhanced hypercube, we have $Q_{n,k} = K_2 \times Q_{n-1,k}$ when $1 \leq k \leq n - 1$. Thus $Q_{k,k}$ is the folded hypercube FQ_k , and $Q_{n,k}$ ($n \geq k + 1$) is built from two copies of $Q_{n-1,k}$ as follows: Let $x \in \{0, 1\}$ and let $xQ_{n-1,k}$ denote the graph obtained by prefixing the label of each node of one copy of $Q_{n-1,k}$ with x ; connect each node $0u_{n-1} \dots u_2 u_1$ of $0Q_{n-1,k}$ with the node $1u_{n-1} \dots u_2 u_1$ of $1Q_{n-1,k}$ by an edge.

3 Cycles embedding in $Q_{n,k}$

For convenience, we use $0G$ and $1G$ to denote the two graphs isomorphism to G , respectively. Then, $K_2 \times G$ can be obtained from $0G$ and $1G$ by joining $0u$ and $1u$

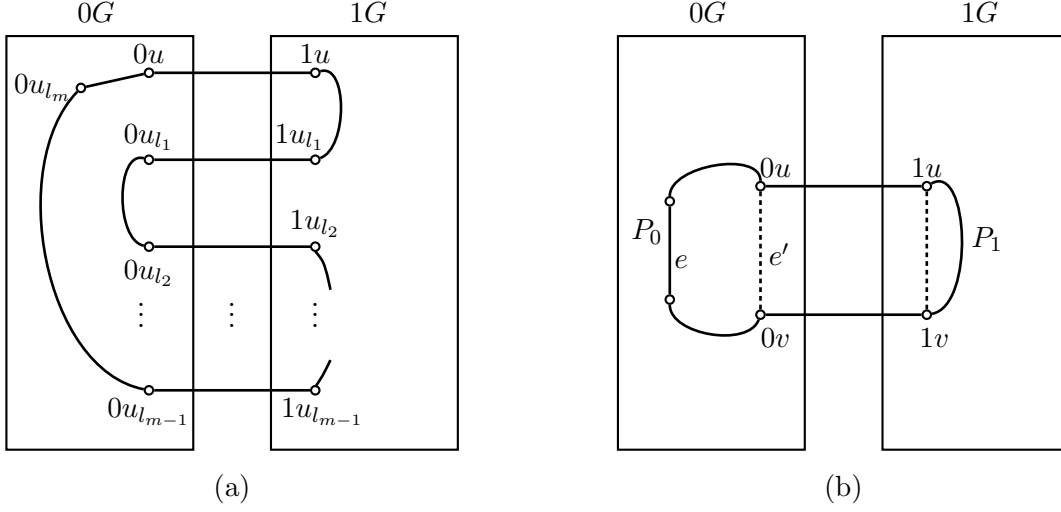


Figure 2: Illustrations for the proof of Lemma 3.1 and 3.3. (A straight or dashed line represents an edge, a curved line represents a path between two vertices.)

between $0G$ and $1G$ for any $u \in V(G)$. The edges between $0G$ and $1G$ are called crossing edges.

We first prove some lemmas about the graph $K_2 \times G$.

Lemma 3.1 *If the length of a shortest odd cycle in a graph G is l , the length of a shortest odd cycle containing a crossing edge is at least $l + 2$ in $K_2 \times G$.*

Proof. Assume the crossing edge is $(0u, 1u)$ in $K_2 \times G$.

We will prove the length of any odd cycle containing the edge $(0u, 1u)$ is at least $l + 2$. Assume the odd cycle is $C = \langle 0u, 1u, 1u_1, \dots, 1u_{l_1}, 0u_{l_1}, 0u_{l_1+1}, \dots, 0u_{l_2}, 1u_{l_2}, \dots, 1u_{l_3}, 0u_{l_3}, \dots, 0u_{l_m}, 0u \rangle$ (See Fig.2(a)). Then the length of the cycle C is $l_m + 1 + m$ and $m \geq 2$ is an even integer. A closed walk W in G is obtained by removing the left bits of the vertices on C . The closed walk W is $\langle u, u_1, \dots, u_{l_1}, \dots, u_{l_2}, \dots, u_{l_3}, \dots, u_{l_m}, u \rangle$. The length of the closed walk W is $l_m + 1$. Since $l_m + 1 + m$ is odd and m is even, $l_m + 1$ is odd. Since any odd closed walk must contain an odd cycle, and the length of any odd cycle in G is at least l , we have $l_m + 1 \geq l$. Hence $l_m + 1 + m \geq l + 2$, that is the length of the odd cycle containing the edge $(0u, 1u)$ is at least $l + 2$. \square

From Lemma 3.1, the following conclusion is obvious.

Corollary 3.2 *If the length of a shortest odd cycle in graph G is l , then the length of a shortest odd cycle in $K_2 \times G$ is also l .*

Lemma 3.3 *If every edge of a connected graph G lies on a cycle of length l for any $l \in I$ (I is a set of integers), then every edge of $K_2 \times G$ lies on a cycle of length l' where $l' \in \{l + 2 | l \in I\} \cup \{l_1 + l_2 | l_1, l_2 \in I\}$.*

Proof. The graph $K_2 \times G$ can be obtained from $0G$ and $1G$ by adding crossing edges. Consider the following two cases.

Case 1 The edge e lies in a subgraph $0G$ or $1G$. Assume e lies in $0G$.

If $l' = l + 2$, there is a cycle C_0 of length l containing e in $0G$. Since $3 \leq l$, there is an edge $e' = (0u, 0v)$ different with e on the cycle C_0 . Let P_0 be the path obtained by deleting the edge e' from C_0 . Then $C = P_0 + \langle 0u, 1u, 1v, 0v \rangle$ is a cycle of length $l' + 2$ containing e in $K_2 \times G$.

If $l' = l_1 + l_2$, there is a cycle C_0 of length l_1 containing e in $0G$. Since $3 \leq l$, there is an edge $e' = (0u, 0v)$ different with e on the cycle C_0 . Let P_0 be the path obtained by deleting the edge e' from C_0 . There is a cycle C_1 of length l_2 in $1G$ containing the edge $(1u, 1v)$. Let P_1 be the path obtained by deleting the edge $(1u, 1v)$ from C_1 . Then $C = P_0 + (0u, 1u) + P_1 + (1v, 0v)$ is a cycle of length l containing e (See Fig.2(b)).

Case 2 The edge $e = (0u, 1u)$ is a crossing edge of $K_2 \times G$.

There is a neighbor $0v$ of $0u$ in $0G$, and $1v$ and $1u$ are also adjacent in $1G$. Then $C = \langle 0u, 1u, 1v, 0v, 0u \rangle$ is a cycle of length 4 containing e .

If $l' = l + 2$, there is a cycle C_0 of length l containing the edge $(0u, 0v)$ in $0G$. Let P_0 be the path obtained by deleting the edge $(0u, 0v)$ from C_0 . Then $C = P_0 + \langle 0u, 1u, 1v, 0v \rangle$ is a cycle of length $l' + 2$ containing $e = (0u, 1u)$ in $K_2 \times G$.

If $l' = l_1 + l_2$, there is a cycle C_0 of length l_1 containing the edge $(0u, 0v)$ in $0G$. Let P_0 be the path obtained by deleting the edge $(0u, 0v)$ from C_0 . There is a cycle C_1 of length l_2 in $1G$ containing the edge $(1u, 1v)$. Let P_1 be the path obtained by deleting the edge $(1u, 1v)$ from C_1 . Then $C = P_0 + (0u, 1u) + P_1 + (1v, 0v)$ is a cycle of length l containing $e = (0u, 1u)$.

The lemma follows. □

The following two lemmas which obtained in the literature make the proofs of our results simple.

Lemma 3.4 ([9]) *Every edge of Q_n lies on a cycle of every even length from 4 to 2^n for $n \geq 2$.*

Lemma 3.5 ([21]) *The n -dimensional folded hypercube FQ_n is bipartite if and only if n is odd, and the minimum length of odd cycles is $n + 1$ if n is even. Every edge of FQ_n lies on a cycle of every even length from 4 to 2^n for $n \geq 3$. Moreover, every edge also lies on a cycle of every odd length from $n + 1$ to $2^n - 1$ if n is even.*

Lemma 3.6 *The enhanced hypercube $Q_{n,k}$ is a bipartite graph if and only if k is odd.*

Proof. Since $Q_{n,k}$ is obtained from Q_n by adding 2^{n-1} skips, to prove the lemma, it is sufficient to consider skips. Let $\{X, Y\}$ be a bipartition of Q_n . Since any vertex $u = u_n \dots u_2 u_1$ and $v = u_n \dots u_{k+1} \bar{u}_k \bar{u}_{k-1} \dots \bar{u}_2 \bar{u}_1$ in Q_n have different parity if and only if k is odd and, hence, any skip in $Q_{n,k}$ joins two vertices in different parts of $\{X, Y\}$. It follows that $Q_{n,k}$ is a bipartite graph if and only if k is odd. \square

Theorem 3.7 *Every edge of $Q_{n,k}$ ($1 \leq k \leq n$) lies on a cycle of every even length from 4 to 2^n for $n \geq 3$. Moreover, every edge lies on a cycle of every odd length from $k+3$ to $2^n - 1$ if k is even, and the edge not in E_i ($k+1 \leq i \leq n$) also lies on an odd cycle of length $k+1$.*

Proof. By Lemmas 3.4 and 3.5, the conclusion is true for $Q_{n,1} = Q_n$ and $Q_{n,n} = FQ_n$. Assume $2 \leq k \leq n-1$. We prove the theorem by induction on $n-k$.

When $n-k=1$ and $n \geq 3$, that is $k=n-1 \geq 2$. We have $Q_{n,n-1} = K_2 \times FQ_n$. Note that FQ_2 is a complete graph K_4 . By Lemmas 3.3 and 3.5, the conclusion is true.

Assume that the conclusion is true for a integer t with $1 \leq n-k < t$. We will prove the case $n-k=t$. Note that $Q_{n,k} = K_2 \times Q_{n-1,k}$.

By the induction hypothesis and Lemma 3.3, every edge of $Q_{n,k}$ lies on a cycle of even length l where $4 \leq l \leq 2^n$.

Assume k is even in the following. By Corollary 3.2 and Lemma 3.5, the length of the shortest odd cycle in $Q_{n,k}$ is $k+1$.

If the edge $e = (0u, 1u)$ is in E_n , that is e is a crossing edge. By the induction hypothesis and Lemma 3.1, the length of the shortest odd cycle containing a crossing edge e is at least $k+3$. Since $k+3 = 2 + (k+1)$, and there is a 1-dimensional edge $(0u, 0v)$ incident with $0u$ in $0Q_{n-1,k}$. By the induction hypothesis, there is a cycle C_0 of length $k+1$ containing the edge $(0u, 0v)$ in $0Q_{n-1,k}$. Let P_0 be the path obtained by deleting the edge $(0u, 0v)$ from C_0 . Then $C = P_0 + \langle 0u, 1u, 1v, 0v \rangle$ is a cycle of length $k+3$ containing $e = (0u, 1u)$ in $Q_{n,k}$. For any odd integer $k+5 \leq l \leq 2^n - 1$, we can express $l = l_1 + l_2$ where $k+1 \leq l_1 \leq 2^{n-1} - 1$ is odd, and $4 \leq l_2 \leq 2^{n-1}$ is even. By Lemma 3.3, there is an odd cycle of length l containing e in $Q_{n,k}$.

If the edge e is not in E_n , that is e is not a crossing edge. We may assume that e is in $0Q_{n-1,k}$. By the induction hypothesis, the edge e lies on an odd cycle of length l ($k+3 \leq l \leq 2^{n-1} - 1$), and the edge e also lies on an odd cycle of length $k+1$ if $e \notin E_i$ ($k+1 \leq i \leq n-1$). For any odd integer $2^{n-1} + 1 \leq l \leq 2^n - 1$, we can express $l = l_1 + l_2$ where $k+3 \leq l_1 \leq 2^{n-1} - 1$ is odd, and $4 \leq l_2 \leq 2^{n-1}$ is even. By Lemma 3.3, there is an odd cycle of length l containing e in $Q_{n,k}$. \square

By Lemma 3.1, Corollary 3.2, and Theorem 3.7, we conclude that the length of a shortest odd cycle is $k + 1$, and the length of a shortest odd cycle containing the edge of dimension i ($k + 1 \leq i \leq n$) is $k + 3$ in $Q_{n,k}$ when k is even. Hence, the results are best possible.

Since every vertex of $Q_{n,k}$ is incident with a unique edge in E_i where $1 \leq i \leq n$, we obtain the following corollary.

Corollary 3.8 *Every vertex of $Q_{n,k}$ lies on a cycle of every even length from 4 to 2^n ; if k is even, every vertex also lies on a cycle of every odd length from $k + 1$ to $2^n - 1$.*

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